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Wang's Proportional Hazard Transform

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CAPM and Mean-Variance: Why?

- CAPM and mean-variance analyses are based on one of two standard assumptions:
    - The mean and the variance determine preferences, or
    - All returns are normally distributed.
- Since the normal is determined by the mean and the variance the first assumption is a generalization of the second.
- In general it is not true that the moments determine a distribution. You can add wiggles to a lognormal or Weibull distribution and leave all the moments unchanged.
- Derivations of the CAPM often start with some statement like: "let $U(m,v)$ be the utility function, written as a function of the mean return m and the variance of return v."
- For quadratic utility functions the mean and variance determine the utility. For non-quadratic utility functions the mean and variance are sometimes used to estimate the utility using a Taylor series argument.
- For applications to stocks, return distributions are approximately normal and are symmetric. Therefore the CAPM assumptions are reasonable.
- The important point is: **mean-variance results are dependent on an ASSUMPTION that the variance is the only measure of risk that concerns the investor**.

Mean-Variance and Insurance

- For insurance applications returns are not normally distributed
- In assessing insurance transactions we are concerned with more than the mean and the variance (cf. FASB 113 and cat insurance). At the very least we are concerned with skewness.
- CAPM does not distinguish between risk (process risk) and uncertainty (parameter risk). Rather it divides risk into systematic (non-diversifiable) and non-systematic (diversifiable) risk. Uncertainty is a large part of insurance.
- Conclude: CAPM does not really apply to insurance

- Failure of CAPM seen in industry pricing of cat business; cat losses are pure unsystematic risk and yet are priced with a considerable risk premium.

- Mean-variance based approaches in the literature include: Meyers' CME model for ILFs, Feldblum's correlations between lines approach, Kreps' marginal surplus and marginal variance approach

- Lane says in *A year of Structuring Furiously*: "Although the differences in the various alternatives is reflected in some standard statistics (standard deviation, probability of loss), they seem inadequate to the task. New measures are needed."

- Philosophically, we want something that involves the whole probability distribution of return, rather than collapsing it to one or two numbers such as the mean and variance.

Mean-Variance-Skewness

- It is possible to derive a mean-variance-skewness version of the CAPM, starting with the assumption that utility is $U(m,v,s)$ rather than $U(m,v)$. The result is a similar formula involving the usual beta plus a beta equivalent for skewness.

- Since most assets have zero skewness they would either not be charged or would receive a small credit from the skewness adjustment

- Insurance products, being heavily skewed, would receive a substantial surcharge. Cat products would receive a very large penalty because of their skewness.

- Junk bond issues, with a high probability of default, have a skewed return distribution, and hence are also surcharged

- Model fits facts observed in market and should be considered for future development

- Utility theory predicts that individuals are variance averse, but prefer (positive) skewness in the return distribution. Insurance products have a negatively skewed return distribution because they have a large downside potential but limited upside potential.

Desirable Properties of a Premium Calculation Method

- A **premium calculation method** is a way of going from a loss distribution to a premium (ignore expenses); it loads the loss cost for risk and profit.

- Example: expected loss + 5% margin—industry standard until 1970!

- Example: expected loss + percent of standard deviation—often used in cat business

- Desirable properties of a premium calculation method include:

    1. Positive loading: risk loading is always positive

    2. No rip-off: risk loaded premium is less than the maximum loss

    3. Sub-additive: risk loaded premium for two risks can be no more than the sum of the risk loaded premiums for each separately

4. Additive in layers: layering a risk should not change the total risk loaded premium

5. Preserve riskiness: a more “risky” exposure should receive a greater risk loading

6. Simple and practical

- Sub-additivity is in-line with risks getting cheaper as they are diversified
- Variance principle is additive for independent risks but super-additive in general
- Variance principle charges more for combining layers since the layers are correlated, this is in contradistinction to 4 (see Meyers CME model for ILF risks).
- If 4 does not hold then there is an arbitrage opportunity in the reinsurance markets: write whole layers and retrocede in horizontal slices. But, there may be significant transaction costs, and the market is not efficient very efficient.

Proportional Hazard Transforms

Idea and Definition

- Idea is to change the probabilities to make bad outcomes more likely and good outcomes less likely
- Premium calculation method then computes expected values with respect to the new probabilities
- Example: to compute ILF's compute limited expected values with respect to adjusted probabilities and take ratios as usual. Risk load is automatically built in.
- Example: to price a cat treaty, adjust CATMAP probabilities (see example below) and compute expected value.
- **Definition of PH-transform**: let X be a loss random variable (or return) and let $S(x) = P(X > x) = 1-F(x)$ be the survivor function. Define a new random variable X' by $P(X' > x) := S'(x) = (S(x))^{1/\rho}$ where $\rho > 1$ is a constant. Then X' is the PH-transform of X.
- Example: X is Exponential with parameter m, so $S(x)=\exp(mx)$ and $S'(x) = \exp(mx/\rho)$. Thus the PH-transform is also exponential with parameter m/ρ.
- Example: X is Pareto with parameters B and Q, so $S(x) = \left(\frac{B}{B+x}\right)^Q$ and $S'(x) = \left(\frac{B}{B+x}\right)^{Q/\rho}$. Thus the PH-transform is also Pareto with parameters B and Q/ρ.
- Example: PH-transforms of Weibull and Burr are again Weibull and Burr
- Example: PH-transform of a lognormal is not lognormal and has not easily computable form. However, it is easy to simulate from.
- Example: if X is the output of a CATMAP run then the probabilities can be adjusted as below. Only 20 events are shown for simplicity. The table shows the derivation of adjusted probabilities. The

expected value is 1,698; the risk adjusted expected value with $\rho = 1.2$ is 2,314.

Prob	Cum Prob	Loss	1−Prob	$(1-\text{Prob})^{1/\rho}$	Adj Prob
5.0%	5%	2	95%	95.8%	4.2%
5.0%	10%	3	90%	91.6%	4.3%
5.0%	15%	4	85%	87.3%	4.3%
5.0%	20%	7	80%	83.0%	4.3%
5.0%	25%	12	75%	78.7%	4.4%
5.0%	30%	20	70%	74.3%	4.4%
5.0%	35%	33	65%	69.8%	4.5%
5.0%	40%	55	60%	65.3%	4.6%
5.0%	45%	90	55%	60.8%	4.6%
5.0%	50%	148	50%	56.1%	4.7%
5.0%	55%	245	45%	51.4%	4.8%
5.0%	60%	403	40%	46.6%	4.9%
5.0%	65%	665	35%	41.7%	5.0%
5.0%	70%	1,097	30%	36.7%	5.2%
5.0%	75%	1,808	25%	31.5%	5.3%
5.0%	80%	2,981	20%	26.2%	5.6%
5.0%	85%	4,915	15%	20.6%	5.9%
5.0%	90%	8,103	10%	14.7%	6.4%
5.0%	95%	13,360	5%	8.2%	8.2%

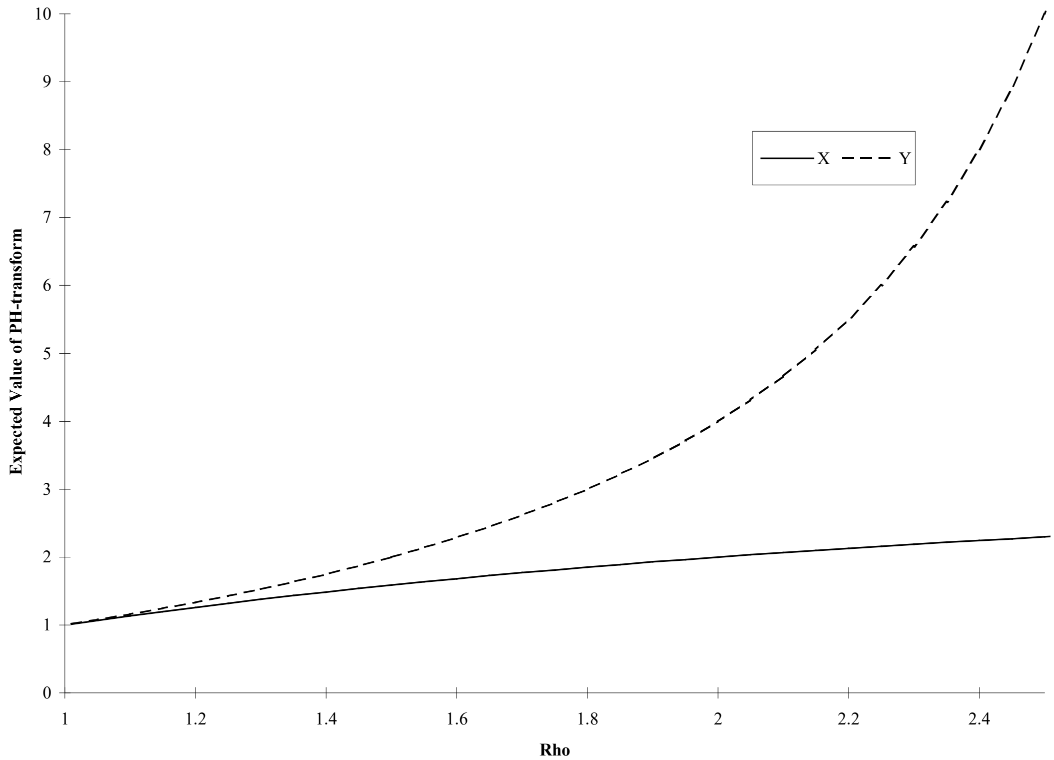
- PH-transform has the desirable properties 1-6 listed above
- Given two loss distributions, neither of which is a hedge against the other, then the PH-transform of the sum is the sum of the PH-transforms. This includes additivity on layers as a special case.
- As the expected loss gets smaller and smaller (e.g. higher and higher attachments) the risk load gets relatively larger and larger. This corresponds to the minimum rate on line phenomenon seen in the market place.

Superiority of PH-transforms to Variance Based Approaches

- Two distributions X and Y
- X has 0.75 chance of no loss and 0.25 chance of a loss of 4
- Y is Pareto, B=2 and Q=3, $S(x) = \left(\frac{2}{2+x}\right)^3$
- X and Y have mean 1
- X and Y have variance 3
- Mean-variance analyses would claim to be indifferent between these two losses! Are *you*? The Pareto has a 3.7% chance of being greater than 4; a 1 in 100 year loss of 7.28 and a 1 in 1000 year loss of 18.
- Expected value of PH-transform of X is $4^{1-1/\rho}$

- Expected value of PH-transform of Y is $\dfrac{2\rho}{3-\rho}$

- Graph below shows the expected values of the PH-transforms of X and Y for different values of ρ



- Expected value of PH-transform of Pareto is greater than X for all $\rho > 1$

- Can show that any smooth utility function will prefer X to the Pareto distribution

Applications

- Can price with one consistent risk load parameter, ρ, across whole book

- Can back into ρ given market pricing to assess relative profitability of different treaties and structures

- Could compare to other financial instruments

Comparison with Other Theories

Utility Theory

- Utility theory is founded on the Morgenstern-von Neumann axioms which imply the existence of a utility function

- Utility function explains preferences, one risky options is preferred to the other if it results in higher expected utility

- Choice of axioms for utility theory is somewhat arbitrary (Wang compares to axioms for Euclidean geometry)

- Yaari and others have proposed a different set of axioms

- Yaari's axioms lead to a theory including PH-transforms as a special case

- PH-transforms are consistent with a utility theory type approach

Option Pricing Theory

- Central tenet of OPT is risk-neutral valuation. Idea is to adjust probabilities to increase likelihood of bad outcomes and decrease likelihood of good outcomes and then price by taking expected values with respect to the new probabilities.

- Black-Scholes: adjusts probabilities so all stocks make the risk free return and then compute the expected value of the option.

- Risk neutral valuation is linked to no-arbitrage pricing methods

- PH-transforms adjust probabilities in the same way and can therefore be regarded as in the same spirit as option pricing theory